4-3 Videos Guide

4-3a

- Taylor series
    - $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

- Maclaurin series
    - $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$

4-3b

- Power series representations of more functions
    - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
    - $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$
    - $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

4-3c

- Taylor polynomials
    - $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$

Theorem (statement):

- Let $R_n(x) = f(x) - T_n(x)$. If $\lim_{n\to\infty} R_n(x) = 0$, then f is equal to the sum of its Taylor series
- Taylor's Inequality
    - If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series of f satisfies $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \le d$.

4-3d

- The Binomial Series
    - $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n + \cdots$

Exercises:

4-3e

- Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a.
  $f(x) = \ln x, \quad a = 1$

4-3f

- Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.
  $f(x) = e^{-2x}$

4-3g

- Use the binomial series to expand the function as a power series. State the radius of convergence.
  $\sqrt[3]{8 + x}$

4-3h

- Use a known Maclaurin series to obtain the Maclaurin series for the given function.
  - $f(x) = e^{3x} - e^{2x}$
  - $f(x) = x^2 \ln(1 + x^3)$
- Use series to evaluate the limit.
  $$\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x}$$
- Find the sum of the series.
  $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$$

4-3i

- Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.
  - $y = e^x \ln(1 + x)$
  - $y = \sec x$

4-3j

- Evaluate the indefinite integral as an infinite series.
  $$\int x^2 \sin(x^2)\ dx$$